A Cluster Expansion for Dipole Gases

Oscar Bolina*
Instituto de Física
Universidade de São Paulo
Caixa Postal 66318
05315-970 São Paulo, Brazil
E-mail: bolina@fma1.if.usp.br

J. Rodrigo Parreira†
Department of Physics
Princeton University
P.O. Box 708
08544-0708 Princeton NJ USA
e-mail: parreira@math.princeton.edu

Abstract

We give a new proof of the well-known upper bound on the correlation function of a gas of non-overlapping dipoles of fixed length and discrete orientation working directly in the charge representation, instead of the more usual sine-Gordon representation.

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1 Introduction

Although dipole gases themselves do not exhibit the Kosterlitz-Thouless transition, they have been used as a natural starting point for the study of this transition in Coulomb systems at low temperatures and small activities [1].

Differently from the behavior of correlations in Coulomb systems, charge correlations in dipole gases have lower and upper bounds with power law decay. The lower bound is a consequence of Jensen's inequality in the charge variables [1]. The main ingredient to obtain an upper bound is the *sine*-Gordon representation of the gas. Using this representation Frohlich and Spencer [2] have

*Supported by FAPESP. Present Address: Department of Mathematics, University of California at Davis, Davis CA 95616-8633, USA

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proved a theorem:

Theorem *The two-point correlation function of external charges for dipoles with hard core is bounded above by*

$$G(x,y) \leq C_\beta e^{-m \log |x-y|} \tag{1.1}$$

with $C_\beta < \infty$ *and*

$$m = \frac{\beta}{2\pi}(1 + (1+\epsilon)\beta z e^{-\beta/16})^{-1}$$

for any $\epsilon > 0$, *where* β *is the inverse temperature and* z *is the activity.*

The correlation function above can be equivalently written as $G(x,y) = e^{-\beta V_{eff}(x,y)}$, where the effective potential $V_{eff}(x,y)$ is the difference between the free energy of the system in the presence of the external charges and the free energy in the absence of the external charges. The physical content of the above theorem is that the long range dependence on the position of the external charges in (1.1) implies that dipoles do not screen [4].

In this paper we dispense with the usual *sine*-Gordon representation and establish the upper bound (1.1) for a gas of non-overlapping dipoles of fixed length and discrete orientation, working directly in the charge (or gas) representation.

To obtain the above result we use an electrostatic inequality in the form of charge translations in order to drive down the activity of the dipoles and obtain estimates for small activities, followed by a direct cluster expansion of the partition function to extract the fall-off of the correlation. (The extension of the technique to Coulomb gases follows along the same lines).

Our method provides an alternative procedure for dealing with systems for which no *sine*-Gordon representation is available. (See [3] for the use of a charge representation in connection with percolation models.) In particular, we hope that the charge representation will enable us to give a direct proof of the analyticity of the pressure both for dipole and Coulomb gases.

2 Dipole Gases

We consider a lattice dipole gas with discrete orientation in a finite region Λ. Let $L = 4r\mathbf{Z}^2$ be a sublattice of Λ of mesh 4r, contained in a box $\hat{\Lambda}$ inside Λ and concentric to it. A configuration of a lattice dipole is described by a function $\vec{\mu}_L = \{\vec{\mu}(j)\}$, where $\vec{\mu}(j) = (\mu_1(j), \mu_2(j))$ is the total dipole

moment at the site $j \in L$. The dipoles are non-overlapping if their fixed length is r. For simplicity we shall take $r = 1$.

The partition function is given by

$$\mathcal{Z} = \sum_{\vec{\mu}_L = \{\vec{\mu}(j), j \in L\}} e^{-\frac{\beta}{2} E(\vec{\mu}_L)} \prod_{j \in L} d\lambda(\vec{\mu}(j)), \tag{2.1}$$

where the interaction energy is

$$E(\vec{\mu}_L) = \sum_{j,k} \sum_{\alpha,\gamma} \mu_\alpha(j) \mu_\gamma(k) W_{\alpha\gamma}(j,k)$$

and the two-body potential is given by

$$W_{\alpha\gamma}(j,k) = \partial_\alpha \partial_\gamma C(j,k)$$

where ∂_α is the usual finite difference derivative in the α-direction, and $C(j,k)$ is the Green's function of a lattice Coulomb potential in two dimensions.

The "a priori" distribution $d\lambda$ of $\vec{\mu}(j)$ is given by the measure

$$d\lambda(\vec{\mu}) = \delta(\vec{\mu}) + z\delta(|\vec{\mu}|^2 - 1)$$

where z is the activity, the same for all the dipoles.

The correlation function is defined by

$$G(x,y) = \frac{\mathcal{Z}_L^\xi(x,y)}{\mathcal{Z}_L},$$

where the $\mathcal{Z}_L^\xi(x,y)$ is the partition function in the presence of the external charges, given by

$$\mathcal{Z}^\xi(x,y) = \sum_{\vec{\mu}_L = \{\vec{\mu}(j), j \in L\}} e^{-\frac{\beta}{2} E(\vec{\mu}_L + \xi(j))} \prod_{j \in L} \lambda(\vec{\mu}(j)).$$

where $\xi(j) = \xi(\delta_{j,x} - \delta_{j,y})$, and $\mathcal{Z}_L = \mathcal{Z}_L^{\xi=0}$.

Throughout this paper we suppose that the external charges are located far apart from each other, and that x and y do not belong to $\hat{\Lambda}$.

3 Electrostatic Inequality

The first step to obtain an upper bound on the correlation function consists in driving down the activity of the dipoles, so as to obtain estimates valid for small values of z. This is done by the

electrostatic inequality. To apply this inequality, it is more convenient to define a dipole charge density by

$$\rho(v) = \sum_{j,\alpha} \mu_\alpha(j)(\delta_{j,v} - \delta_{j,v+e_\alpha})$$

and write the electrostatic energy as $E(\vec{\mu}_L) = E(\rho) = (\rho, -\Delta^{-1}\rho)$. If we carry out the summation on the dipole moment variables in (2.1), we get

$$\mathcal{Z} = \sum_\rho e^{-\frac{\beta}{2}E(\rho)} \prod_{j,\alpha} z_\alpha(j)$$

where $z_\alpha(j) = z(j)^{|\mu_\alpha(j)|}$. Because of the discrete orientation of the dipoles, $\mu_\alpha(j)$ takes the values $\{0, \pm 1\}$ in such a way that $\mu_1(j)\mu_2(j) = 0$.

Now we replace the charge density ρ, whose support lies in $\hat{\Lambda}$, by a charge density $\bar{\rho}$, whose support lies in Λ, in such a way that the interaction energy between the two renormalized densities remain unchanged, but that the self-energy of $\bar{\rho}$ is smaller than the self-energy of ρ. We use the difference in self-energies to renormalize the activities [1].

The renormalized charge density is given by

$$\bar{\rho} = \rho + \frac{\Delta\rho}{||\Delta||}. \tag{3.1}$$

For the purpose of renormalizing the activity of the dipoles, we write

$$\mathcal{Z} = \sum_{\bar{\rho}} e^{-\frac{\beta}{2}E(\bar{\rho})} e^{-\frac{\beta}{2}[E(\rho)-E(\bar{\rho})]} \prod_{j,\alpha} z_\alpha(j). \tag{3.2}$$

The difference in self-energy is estimated as follows:

$$E(\rho) - E(\bar{\rho}) = \frac{2}{||\Delta||}(\rho, \rho) - \frac{1}{||\Delta||^2}(\rho, -\Delta\rho) \geq \frac{1}{||\Delta||}(\rho, \rho).$$

So that we obtain

$$\mathcal{Z} = \sum_{\bar{\rho}} e^{-\frac{\beta}{2}E(\bar{\rho})} \prod_{j,\alpha} \bar{z}_\alpha(j), \tag{3.3}$$

with the estimate $\bar{z}_\alpha(j) \leq z_\alpha(j) \exp\{-(\beta/||\Delta||)\rho_\alpha^2(j)\}$.

Since the activities are all equal, all the renormalized activities can be bounded by the same constant (take $||\Delta|| = 8$) :

$$\bar{z}_\alpha(j) \leq \bar{z} = z \exp\{-\frac{\beta}{16}\}, \tag{3.4}$$

which can be made arbitrarily small for β sufficiently large.

4 The External Charges

We apply the above reasoning to the partition function of the system in the presence of external charges, and obtain

$$\mathcal{Z}^{\xi}(x,y) = \sum_{\bar{\rho}} e^{-\frac{\beta}{2}E(\bar{\rho}+\xi)} \prod_{j,\alpha} \bar{z}_{\alpha}(j).$$

However, in this case we will need to renormalize the external charges as well. This is done by the shift

$$\sigma = \bar{\rho} + \xi + \frac{\gamma}{\beta}(\Delta a)(j)$$

where γ depends on β in a way to be specified later, and

$$a(j) = C(j,x) - C(j,y). \tag{4.1}$$

The function $a(j)$ satisfies, for large $|x-y|$ and for $|j-j'|=1$, the bounds [3, chapter 7]:

$$|a(j) - a(j')| \leq \text{const}\left(\frac{1}{|j-x|+1} + \frac{1}{|j-y|+1}\right), \tag{4.2}$$

$$|a(j) - a(j')| \leq \text{const}\frac{|x-y|}{|j|^2}, \tag{4.3}$$

From the definition of the Laplacian operator and (4.1), we also have

$$\sum_{|j-j'|=1} [a(j) - a(j')]^2 \leq (a, -\Delta a) = a(x) - a(y) \approx \frac{1}{\pi}\log|x-y|. \tag{4.4}$$

After the shift we get, following Eqs. (3.2) and (3.3),

$$\mathcal{Z}^{\xi}(x,y) = e^{(\frac{\gamma^2}{2\beta}-\gamma)(a,-\Delta a)} Z_1, \tag{4.5}$$

where

$$Z_1 = \sum_{\sigma} e^{-\frac{\beta}{2}E(\sigma)} \prod_{j,\alpha} \bar{z}_{\alpha}(j) e^{\gamma_{\alpha}(j)}, \tag{4.6}$$

with

$$\gamma_{\alpha}(j) = \gamma\mu_{\alpha}(j)\delta_{\alpha}\bar{a}(j), \tag{4.7}$$

and the definition of $\bar{a}(j)$ follows from (3.1).

5 The Cluster Expansion

To derive an upper bound on the partition function (4.6) we write

$$\prod_{j,\alpha} \bar{z}_\alpha(j) e^{\gamma_\alpha(j)} = \prod_{j,\alpha} \bar{z}_\alpha(j) \cosh \gamma_\alpha(j) + R$$

where ($\mu = 1, 2$)

$$R = \prod_{j,\alpha} (\bar{z}_\alpha(j) \cosh \gamma_\alpha(j) + \bar{z}_\alpha(j) \sinh \gamma_\alpha(j)) - \prod_{j,\alpha} \bar{z}_\alpha(j) \cosh \gamma_\alpha(j) \tag{5.1}$$

Thus (4.6) becomes

$$Z_1 = \sum_\sigma e^{-\frac{\beta}{2} E(\sigma)} \left\{ \prod_{j,\alpha} \bar{z}_\alpha(j) \cosh \gamma_\alpha(j) + R \right\} \tag{5.2}$$

The function $\gamma_\alpha(j)$ depends on $\delta_\alpha \bar{a}(j)$. However, note that

$$\delta_\alpha \bar{a}(j) = \bar{a}(j) - a(j+e) = a(j) - a(j+e) \quad \text{when} \quad j \neq x, y,$$

since $(\Delta a)(j) = 0$ for $j \neq x, y$.

From the definition (4.7), and the bounds (4.2), (4.3) we see that the argument of $\cosh(\cdot)$ is small for $|j - x| \gg \gamma$ and $|j - y| \gg \gamma$. Thus we can write

$$\cosh \gamma_\alpha(j) - 1 \leq \frac{1}{2}(1 + \epsilon)\gamma^2 [a(j) - a(j')]^2$$

for some $\epsilon > 0$.

For $|j - x| \leq O(\gamma), |j - y| \leq O(\gamma)$, we estimate $\cosh(\cdot) - 1$ by a constant.

In both cases we can take z sufficiently small so that the (5.1) term can be bounded by a constant for large β.

Under these circumstances, the R-terms in (5.2) can be neglected:

$$Z_1 \leq K_\beta \sum_\sigma e^{-\frac{\beta}{2} E(\sigma)} \prod_{j,\alpha} \bar{z}_\alpha(j) \cosh \gamma_\alpha(j) \tag{5.3}$$

for some positive constant K_β.

Now we develop a cluster expansion for $\prod_{j,\alpha} \cosh \gamma_\alpha(j)$ as follows

$$\prod_{j,\alpha} [1 + (\cosh \gamma_\alpha(j) - 1)] = 1 + S_1 + S_2 + S_1 S_2 \tag{5.4}$$

where

$$S_\alpha = \sum_{m_\alpha \geq 1} \sum_{j_1,\ldots,j_{m_\alpha}} D_\alpha(j_1). \;\; . \;\; .D_\alpha(j_{m_\alpha}), \tag{5.5}$$

with

$$D_\alpha(j_i) = \cosh \gamma_\alpha(j_i) - 1. \tag{5.6}$$

The sum in (5.5) above runs over all subsequences of $\{j_1, \ldots j_{m_\alpha}\}$, $m \geq 1$.

We can distribute the activities inside S_α, to get

$$Z_1 \leq K_\beta \left\{ \mathcal{Z} + \sum_\sigma e^{-\frac{\beta}{2}E(\sigma)}(s_1 + s_2 + s_1 s_2) \right\}, \tag{5.7}$$

where $s = s_1 + s_2 + s_1 s_2$ and, as in (5.5),

$$s_\alpha = \sum_{m_\alpha \geq 1} \sum_{j_1,\ldots,j_{m_\alpha}} \bar{z}_\alpha(j_1) D_\alpha(j_1). \;\; . \;\; .\bar{z}_\alpha(j_{m_\alpha}) D_\alpha(j_{m_\alpha}).$$

Thus we see that Z_1 admits the upper bound

$$Z_1 \leq K_\beta \mathcal{Z}(1 + s_1 + s_2 + s_1 s_2). \tag{5.8}$$

Here the s terms in (5.7) have been taken with $\mu_\alpha(j) = 1$ for all j, α, and we have used that $\mathcal{Z} > 1$.

If we *undo* the cluster expansion (5.4) we obtain

$$Z_1 \leq K_\beta \mathcal{Z} \prod_{j,\alpha} (1 + \bar{z}_\alpha(j) D_\alpha(j)), \tag{5.9}$$

with $\mu_\alpha(j) = 1$ for all j, α.

Substituting (5.9) in (4.6) and using (3.4) and (4.4), we get

$$\mathcal{Z}^\xi(x, y) \leq K_\beta \, \mathcal{Z} \, e^{\{\frac{\gamma^2}{2\beta} + \frac{1}{2}(1+\epsilon)\bar{z}\gamma^2 - \gamma\}(a, -\Delta a)}. \tag{5.10}$$

The optimal choice for γ is $\gamma = \beta(1 + (1 + \epsilon)\bar{z}\beta))^{-1}$, for which we finally get

$$G(x, y) \leq K_\beta e^{-\frac{\beta}{2\pi} \frac{1}{1+(1+\epsilon)\bar{z}\beta} \log|x-y|},$$

which concludes the argument.

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